

BOOTSTRAPPED MORAWETZ ESTIMATES AND RESONANT DECOMPOSITION FOR LOW REGULARITY GLOBAL SOLUTIONS OF CUBIC NLS ON \mathbb{R}^2

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ABSTRACT. We prove global well-posedness for the L^2 -critical cubic defocusing nonlinear Schrödinger equation on \mathbb{R}^2 with data $u_0 \in H^s(\mathbb{R}^2)$ for $s > \frac{1}{3}$. The proof combines *a priori* Morawetz estimates obtained in [4] and the improved almost conservation law obtained in [6]. There are two technical difficulties. The first one is to estimate the variation of the improved almost conservation law on intervals given in terms of Strichartz spaces rather than in terms of $X^{s,b}$ spaces. The second one is to control the error of the *a priori* Morawetz estimates on an arbitrary large time interval, which is performed by a bootstrap via a double layer in time decomposition.

1. INTRODUCTION

We shall consider the L^2 -critical Schrödinger equation on \mathbb{R}^2

$$(1.1) \quad iu_t + \Delta u = |u|^2 u$$

with data $u(0) = u_0 \in H^s(\mathbb{R}^2)$, $s \geq 0$. Here $H^s(\mathbb{R}^2)$ denotes the Sobolev space endowed with the norm

$$(1.2) \quad \|f\|_{H^s(\mathbb{R}^2)} := \| \langle \xi \rangle^s \widehat{f}(\xi) \|_{L^2(\mathbb{R}^2)}$$

with \widehat{f} denoting the Fourier transform

$$(1.3) \quad \widehat{f}(\xi) := \int_{\mathbb{R}^2} f(x) e^{-ix \cdot \xi} dx$$

and $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. This problem is known to be locally well-posed [3] for any $s \geq 0$. If $s > 0$ then local well-posedness means that for any data $u_0 \in H^s(\mathbb{R}^2)$, there exists a time of local existence $T_l = T_l(\|u_0\|_{H^s(\mathbb{R}^2)})$ depending only on the norm of the initial data and a unique solution u lying in a Banach space $X \subset \mathcal{C}([0, T_l], H^s(\mathbb{R}^2))$ such that $u(t)$ satisfies for $t \in [0, T_l]$ the Duhamel formula

$$(1.4) \quad u(t) := e^{it\Delta} u_0 - i \int_0^t e^{i(t-t')\Delta} \left[|u|^2 u(t') \right] dt'$$

and the solution depends continuously on the norm of the initial data. Local-in-time H^s -solutions to (1.1) satisfy the *mass conservation law*

$$(1.5) \quad \|u(t)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}$$

and local-in-time H^1 -solutions to (1.1) satisfy the *energy conservation law*

$$(1.6) \quad E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} |u(t, x)|^4 dx.$$

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In this paper we are interested in proving that H^s -solutions to (1.1) with $s \geq 0$ exist for all time $T \geq 0$. If $s > 0$ then in view of the local well-posedness theory it suffices to find an *a priori* bound of the form

$$(1.7) \quad \|u(T)\|_{H^s(\mathbb{R}^2)} \leq Q(\|u_0\|_{H^s(\mathbb{R}^2)}, T)$$

with Q a function depending only on the norm of the initial data and time T . If $s = 1$ then the energy conservation law immediately yields the bound (1.7). No blowup solutions are known for (1.1). It is conjectured that (1.1) is globally well-posed in $H^s(\mathbb{R}^2)$ for $1 > s \geq 0$. The first breakthrough to establish global well-posedness below the energy threshold, by using what is now referred to as the *Fourier truncation method*, appears in [1]. He showed global well-posedness for data in $H^s(\mathbb{R}^2)$ with $s > \frac{3}{5}$. A sequence of works ([5, 6, 7, 4]) has lowered the regularity requirements for global well-posedness for (1.1) down to $s > \frac{2}{5}$. Recently, the conjecture was proved in [8], in the case of spherically symmetric initial data. The main result of this paper is the following improvement:

Theorem 1.1. *The L^2 -critical Schrödinger equation on \mathbb{R}^2 is globally well-posed in $H^s(\mathbb{R}^2)$, $1 > s > \frac{1}{3}$. Moreover there exists a constant C depending only on $\|u_0\|_{H^s(\mathbb{R}^2)}$ such that*

$$(1.8) \quad \|u(T)\|_{H^s(\mathbb{R}^2)}^2 \leq C(\|u_0\|_{H^s(\mathbb{R}^2)})T^{\frac{1-s}{3s-1}+}$$

for all times T .

Before sketching the main ideas underpinning this theorem, we set up some notation.

Given A, B two nonnegative numbers, $A \lesssim B$ means that there exists a universal nonnegative constant K such that $A \leq KB$. We say that K_0 is the constant determined by the relation $A \lesssim B$ if K_0 is the smallest K such that $A \leq KB$ is true. We write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$. $A << B$ denotes $A \leq KB$ for some universal constant $K < \frac{1}{100}$. We also use the notations $A+ = A + \epsilon$, $A++ = A + 2\epsilon$, $A- = A - \epsilon$ and $A-- = A - 2\epsilon$, etc. for some universal constant $0 < \epsilon \ll 1$. We shall abuse the notation and write $+$, $-$ for $0+$, $0-$ respectively.

Let $\lambda \in \mathbb{R}$ and let u^λ denote the following function

$$(1.9) \quad u^\lambda(t, x) := \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right).$$

We recall that if u satisfies (1.1) with data u_0 then u^λ also satisfies (1.1) but with data $\frac{1}{\lambda}u_0\left(\frac{x}{\lambda}\right)$.

If $J := [a, b]$ is an interval then $|J|$ is its size. A partition $\mathcal{P}_\mu(J) = (J_i)_{i \in \{1, \dots, l\}}$ of a finite interval J is of size μ , $\mu > 0$ if three conditions are satisfied

- (1) $\bigcup_{i \in \{1, \dots, l\}} J_i = J$
- (2) $J_i \cap J_j = \emptyset, i \neq j$
- (3) $|J_i| = \mu, i \in \{1, \dots, l-1\}$.

If u is a solution of (1.1) on J then we can write u as the sum of its linear part and its nonlinear part; more precisely

$$(1.10) \quad u(t) = u_J^l(t) + u_J^{nl}(t)$$

with

$$(1.11) \quad u_J^l(t) := e^{i(t-a)\Delta} u(a)$$

and

$$(1.12) \quad u_J^{nl}(t) := -i \int_a^t e^{i(t-t')\Delta} \left[|u|^2 u(t') \right] dt'.$$

Let dt denote the standard Lebesgue measure and let $d\mu_l$ ¹ be the following measure

$$(1.13) \quad d\mu_l := \delta(t - a)dt.$$

If $(p, q) \in [1, \infty]$ then we define the spaces $L^p(J)$ and $L^p(J, d\mu^l)$

$$(1.14) \quad \begin{aligned} L^p(J) &:= \{f : \mathbb{R} \rightarrow \mathbb{C}, \|f\|_{L^p}^p := \int_J |f|^p dt < \infty\} \\ L^p(J, d\mu^l) &:= \{f : \mathbb{R} \rightarrow \mathbb{C}, \|f\|_{L^p(d\mu^l)}^p := \int_J |f|^p d\mu_l < \infty\} \end{aligned}$$

and the mixed spaces

$$(1.15) \quad \begin{aligned} L_t^p(J)L_x^q &:= \left\{f : \mathbb{R}^{2+1} \rightarrow \mathbb{C}, \|f\|_{L_t^p(J)L_x^q}^p := \int_J \left(\int_{\mathbb{R}^2} |f(t, x)|^q dx \right)^{\frac{q}{p}} dt < \infty \right\} \\ L_t^p(J, d\mu_l)L_x^q &:= \left\{f : \mathbb{R}^{2+1} \rightarrow \mathbb{C}, \|f\|_{L_t^p(J, d\mu_l)L_x^q}^p := \int_J \left(\int_{\mathbb{R}^2} |f(t, x)|^q dx \right)^{\frac{q}{p}} d\mu_l < \infty \right\} \end{aligned}$$

Let \tilde{f} be the spacetime Fourier transform of a function f

$$(1.16) \quad \tilde{f}(t, x) := \int_{\mathbb{R}^{2+1}} f(t, x) e^{-i(t\tau + x\xi)} dt dx$$

If p is an integer larger or equal to one, $\sigma : \mathbb{R}^{2p} \rightarrow \mathbb{C}$ is a smooth symbol and u_1, \dots, u_{2p} are Schwartz functions then we define the $2p$ -linear functionals

$$(1.17) \quad \Lambda_{2p}(\sigma; u_1(t), \dots, u_{2p}(t)) := \int_{\xi_1 + \dots + \xi_{2p} = 0} \sigma(\xi_1, \dots, \xi_{2p}) \prod_{j \text{ odd}} \widehat{u}_j(t, \xi_j) \prod_{j \text{ even}} \widehat{\bar{u}}_j(t, \xi_j)$$

and

$$(1.18) \quad \Lambda_{2p, J}(\sigma; u_1, \dots, u_{2p}) := \int_J \int_{\xi_1 + \dots + \xi_{2p} = 0} \sigma(\xi_1, \dots, \xi_{2p}) \prod_{j \text{ odd}} \widehat{u}_j(t, \xi_j) \prod_{j \text{ even}} \widehat{\bar{u}}_j(t, \xi_j).$$

If $u_1 = \dots = u_{2p} = u$ then we abbreviate $\Lambda_{2p}(\sigma; u) := \Lambda_{2p}(\sigma; u_1, \dots, u_{2p})$ and $\Lambda_{2p, J}(\sigma; u) := \Lambda_{2p, J}(\sigma; u_1, \dots, u_{2p})$. Let $\Omega_k^{1 \rightarrow 2p}$ denote the set of unordered subsets of size k from the set $\{0, \dots, 2p\}$. If $A \in \Omega_k^{1 \rightarrow 2p}$ then we write $\Lambda_{2p, J, A}(\sigma; u)$ for $\Lambda_{2p, J}(\sigma; v_1, \dots, v_{2p})$ with $v_i = u_j^l$ if $i \in A$ and $v_i = u_j^{nl}$ if $i \notin A$. Let

$$(1.19) \quad L := \bigcap_{i=1}^4 \{(\xi_j)_{j \in [1, \dots, 4]}, |\xi_j| \leq \frac{N}{100}\}$$

and

$$(1.20) \quad \Gamma := \{(\xi_j)_{j \in [1, \dots, 4]}, |\cos(\xi_{12}, \xi_{14})| \geq \theta\}$$

where $0 < \theta/l$ is a parameter to be determined. Here we use the convention $\xi_{ab} := \xi_a + \xi_b$, $\xi_{abc} := \xi_a + \xi_b + \xi_c$, etc.

We constantly use the I -method [5] throughout this paper in order to find a pointwise-in-time upper bound of the H^s -norm of the solution to (1.1) with data $u(0) = u_0 \in H^s(\mathbb{R}^2)$. We recall it now. Let I be the following multiplier

$$(1.21) \quad \widehat{If}(\xi) := m(\xi) \widehat{f}(\xi)$$

where $m(\xi) := \eta\left(\frac{\xi}{N}\right)$, η is a smooth, radial, nonincreasing in $|\xi|$ such that

$$(1.22) \quad \eta(\xi) := \begin{cases} 1, & |\xi| \leq 1 \\ \frac{1}{|\xi|^{1-s}}, & |\xi| \geq 2 \end{cases}$$

¹ μ_l since this measure will be applied to be the linear part u_j^l of u

By plugging the multiplier I into the energy conservation law (1.6) we define the so-called modified energy

$$(1.23) \quad E(Iu(t)) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla Iu(t, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} |Iu(t, x)|^4 dx.$$

The following proposition [5] shows that it suffices to estimate the modified energy at time T in order to find an upper bound of the pointwise-in-time H^s -norm of the solution u to (1.1) with data $u(0) = u_0 \in H^s(\mathbb{R}^2)$; more precisely

Proposition 1.2 ($H^s(\mathbb{R}^2)$ norm and modified energy are comparable [5]). *For all time $T \geq 0$*

$$(1.24) \quad \|u(T)\|_{H^s(\mathbb{R}^2)}^2 \lesssim E(Iu(T)) + \|u_0\|_{H^s(\mathbb{R}^2)}^2.$$

Since the symbol of I approaches one as N goes to infinity we expect the variation of the modified energy to be slower and slower as N increases. Therefore we estimate the modified energy by using the fundamental theorem of calculus and we use Proposition 1.2 to control $\|u(T)\|_{H^s(\mathbb{R}^2)}$.

The paper is organized as follows:

In Section 2 we recall the main ideas of [6]. In particular, we explain their construction of a new almost conservation law $\tilde{E}(u(t))$ which is close to the modified energy $E(Iu(t))$ at each time t and how they estimated the variation of $\tilde{E}(u(t))$ on an interval of size, roughly speaking, equal to one. In Section 3 we recall the main results of [4] and, in particular, the Morawetz-type estimates. We would like to combine the ideas from [6] with those from [4]. However there are two non-trivial difficulties that appear.

The I -method is based upon an estimate of the variation of an almost conservation law on a small interval where we have a control of a large number of norms. Then the variation of the almost conservation law on an arbitrary large time interval $[0, T]$ is estimated by iteration on each subinterval of a partition of $[0, T]$ where this local control holds. This total variation must be controlled at the end of the process. Therefore, if we can establish a local control on a subinterval as large as possible then the number of iterations is reduced and we have a better control of the total variation, which implies global well-posedness for rougher data. Unfortunately we cannot use the result established in [6] (see Proposition 2.2) to estimate the variation of $\tilde{E}(u(t))$ since the local control of the solution in $X^{s,b}$ spaces is only true for short time intervals (see Proposition 2.4). This is due to the nature of these spaces: they describe very well the solution locally in time but not on long time intervals. Proposition 3.1 shows that we have a local control on intervals J where the $L_t^4 L_x^4$ norm of Iu is small. The first idea would be to divide $[0, T]$ into subintervals J where the $L_t^4 L_x^4$ norm of Iu is small. Indeed their size is expected to be, roughly speaking, larger than one, because the Morawetz-type estimates provide good control of the $L_t^4 L_x^4$ norm of Iu on $[0, T]$. In Section 5 we estimate the variation of \tilde{E} on J . The proof has similarities with that of (2.10) but there are differences in the method. We write the variation of \tilde{E} on J in the spacetime Fourier domain, we decompose u into the sum of its linear part u_J^l and its nonlinear part u_J^{nl} , and after some measure rearrangements performed via the use of Fubini's theorem, we use some refined bilinear estimates [1, 6]. These estimates are key estimates to get a slow increase of \tilde{E} . At the end of the process we can bound the variation of \tilde{E} by some quantities that are estimated by the local control theory in turn.

Unfortunately, if we use the Morawetz-type estimate on the whole $[0, T]$ then an error term appears and, as time T goes to infinity, it grows at a faster rate than that generated by the variation of the modified energy on the same interval. The control of the error term is possible if and only if $s > \frac{2}{5}$ (see [4]). We would like to use the Morawetz-type estimates in a better way. To this end we perform a double layer in time decomposition. First we divide $[0, T]$ into subintervals J of size, roughly speaking, equal to N^{3-} . This enables us to control the error term of the Morawetz estimate on J by its main term. Then we decompose each J into subintervals J_k where the $L_t^4 L_x^4$ norm of Iu

is small. By applying the local control theory and the almost conservation law in $L_t^q L_x^r$ spaces (see Proposition 4.1) we can estimate the variation of \tilde{E} on J_k and then on J by iteration. The final step is to bootstrap the Morawetz estimates. More precisely, we use for every J the corresponding Morawetz estimate and we iterate to estimate the variation of \tilde{E} on the whole interval $[0, T]$. At the end of the proof we can control the modified energy on $[0, T]$, provided that $s > \frac{1}{3}$. The whole process is explained in Section 4.

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2. SUMMARY OF [6]

In this section we recall the main ideas and results of [6] since we will often refer to them throughout this paper.

The variation of the modified energy $E(Iu(t))$ is not equal to zero, because of the presence of the commutator $Iu^3 - (Iu)^3$. The control of this variation is possible if the Sobolev exponent s is larger than a threshold s_0 . This control implies global existence for data in $H^s(\mathbb{R}^2)$, $s > s_0$. In [6] the authors aimed at designing a new almost conservation law $\tilde{E}(u(t))$ that would satisfy two properties

- (1) Almost conservation law: $\tilde{E}(u(t))$ would have a slower variation than $E(Iu(t))$
- (2) Proximity to $E(Iu(t))$ at each time t : this property would allow to control $E(Iu(t))$ via $\tilde{E}(u(t))$

To this end they searched for a candidate \tilde{E} that would have the following form

$$(2.1) \quad \tilde{E}(u(t)) := \frac{1}{2}\Lambda_2(\sigma_2; u(t)) + \Lambda_4(\sigma_4; u(t))$$

with σ_2 denoting the following multiplier

$$(2.2) \quad \sigma_2 := -\xi_1 m(\xi_1) \cdot \xi_2 m(\xi_2)$$

and σ_4 to be determined. Notice that $\Lambda_2(\sigma_2)(u(t))$ is nothing else but the kinetic part of the modified energy, i.e $\Lambda_2(\sigma_2)(u(t)) = \frac{1}{2}\|Iu(t)\|_{H^1}^2$. The idea is to substitute the potential term $V(t) := \frac{1}{4} \int_{\mathbb{R}^2} |Iu(t, x)|^4 dx$ of the modified energy $E(Iu)$ for a new quadrilinear term $\Lambda_4(\sigma_4; u(t))$ and to search for some cancellations in the computation of the derivative $\partial_t \tilde{E}(u(t))$. If we compute the derivative of $\Lambda_2(\sigma_2; u(t))$ and $\Lambda_4(\sigma_4; u(t))$ then we find, by using (1.1)

$$(2.3) \quad \partial_t \left(\frac{1}{2} \Lambda_2(\sigma_2; u(t)) \right) : = \Lambda_4(\mu; u(t))$$

and

$$(2.4) \quad \partial_t \Lambda_4(\sigma_4; u(t)) := \Lambda_4(\alpha_4; u(t)) + \Lambda_6(\nu_6; u(t))$$

with

$$(2.5) \quad \mu := \frac{i}{4} (|\xi_1|^2 m^2(\xi_1) - |\xi_2|^2 m^2(\xi_2) + |\xi_3|^2 m^2(\xi_3) - |\xi_4|^2 m^2(\xi_4))$$

$$(2.6) \quad \alpha_4 := -i (|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2)$$

and

$$(2.7) \quad \nu_6 := -i \sum_{k=1}^4 (-1)^{k+1} \sigma_4(\xi_1, \dots, \xi_k + \dots + \xi_{k+2}, \xi_{k+3}, \dots, \xi_6)$$

The authors tried to cancel the quadrilinear terms resulting from the derivative of $\tilde{E}(u(t))$ by letting $\sigma_4 := -\frac{\mu}{\alpha_4}$. The problem is that the singularity $\alpha_4 = 0$ appears. Therefore they had to truncate optimally σ_4 away from $\alpha_4 = 0$ so that the truncation does not totally lose the effect of these cancellations. This requires a detailed study of the singularity. Recall that the corrective term $\Lambda_4(\sigma_4; u(t))$ is a quadrilinear integral evaluated on the convolution surface $\xi_1 + \dots + \xi_4 = 0$. They observed that $\alpha_4 = 2i\xi_{12}\xi_{14} \cos(\xi_{12}, \xi_{14})$ on this surface and that if $|\xi_i| \ll N$, $i \in \{1, \dots, 4\}$ then the singularity disappears and $\alpha_4 = \frac{1}{4}^2$. Therefore they truncated $-\frac{\mu}{\alpha_4}$ in the following way

$$(2.8) \quad \sigma_4(\xi_1, \dots, \xi_4) := -\frac{\mu}{\alpha_4} \chi_{L \cup \Gamma}(\xi_1, \dots, \xi_4)$$

With this value for σ_4 , $\tilde{E}(u(t))$ is well-defined by (2.1). They showed that $\tilde{E}(u(t))$ and $E(Iu(t))$ are closed to each other at each time t ; more precisely

Proposition 2.1 (Proximity to $E(Iu(t))$ at each time t [6]).

$$(2.9) \quad \left| \tilde{E}(u(t)) - E(Iu(t)) \right| \lesssim \frac{1}{\theta N^{2-}} \|Iu(t)\|_{H^1(\mathbb{R}^2)}^4.$$

Then, by using a delicate multilinear analysis, they proved the following result

Proposition 2.2 (Almost Conservation Law in $X^{s,b}$ spaces [6]).

$$(2.10) \quad \left| \sup_{t \in J} \tilde{E}(u(t)) - \tilde{E}(u(a)) \right| \lesssim \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}} + \frac{1}{\theta N^{3-}} \right) \|Iu\|_{X^{1, \frac{1}{2}+}}^4$$

The definition of the $X^{1, \frac{1}{2}+}$ spaces can be found in [2] for example. The proof of Proposition 2.2 extensively relies upon two refined bilinear estimates

Proposition 2.3 (Bilinear estimates [1], [6]). *Let f, g be two Schwartz functions. Let N_1, N_2 be two dyadic numbers such that $N_1 < N_2$. Let θ be a parameter such that $0 < \theta \ll 1$. If*

$$(2.11) \quad B_\epsilon(\tau, \xi) := \int_{\xi_1 + \xi_2 = \xi} \chi_{|\xi_1| \sim N_1} \chi_{|\xi_2| \sim N_2} \chi_{[-\epsilon, \epsilon]}(\tau - |\xi_1|^2 - |\xi_2|^2) \hat{f}(\xi_1) \hat{g}(\xi_2) d\xi_1$$

and

$$(2.12) \quad B_{\epsilon, \theta}(\tau, \xi) := \int_{\xi_1 + \xi_2 = \xi} \chi_{|\xi_1| \sim N_1} \chi_{|\xi_2| \sim N_2} \chi_{|\cos(\xi_1, \xi_2)| \leq \theta} \chi_{[-\epsilon, \epsilon]}(\tau - |\xi_1|^2 - |\xi_2|^2) \hat{f}(\xi_1) \hat{g}(\xi_2) d\xi_1$$

then

$$(2.13) \quad \overline{\lim} \frac{1}{2\epsilon} \|B_\epsilon\|_{L_\tau^2 L_\xi^2} \lesssim \left(\frac{N_1}{N_2} \right)^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}$$

and

$$(2.14) \quad \overline{\lim} \frac{1}{2\epsilon} \|B_{\epsilon, \theta}\|_{L_\tau^2 L_\xi^2} \lesssim \theta^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}.$$

The same conclusions hold if $-|\xi_1|^2 - |\xi_2|^2$ is substituted for $|\xi_1|^2 + |\xi_2|^2$ in (2.11) and (2.12).

Proposition 2.2 shows that we can estimate the variation of $\tilde{E}(u(t))$ on an interval J provided that we can control the $X^{1, \frac{1}{2}+}$ norm of Iu . The next proposition shows that such a control is possible as long as the size of J is, roughly speaking, bounded by one

²This phenomenon is expected. Indeed if all the frequencies have amplitude smaller than N then the modified energy is the energy itself and the variation is equal to zero.

Proposition 2.4 (Modified Local Well-Posedness in $X^{s,b}$ spaces [6]). *There exists $1 < \epsilon \lesssim 1$ such that if $\sup_{t \in J} E(Iu(t)) \lesssim 1$ and $|J| \leq \epsilon$ then*

$$(2.15) \quad \|\eta(t-a)Iu\|_{X^{1, \frac{1}{2}+}} \lesssim 1$$

with η bump function adapted to $[-\epsilon, \epsilon]$.

Finally, by choosing the optimal parameter $\theta = \frac{1}{N}$, they estimated the variation of the almost conservation law \tilde{E} on an interval J of size one

$$(2.16) \quad |\sup_{t \in J} E(Iu(t)) - \tilde{E}(u(a))| \lesssim \frac{1}{N^{2-}}$$

The variation is slower than that of the modified energy. Indeed this $O\left(\frac{1}{N^{2-}}\right)$ increase is smaller than the $O\left(\frac{1}{N^{\frac{3}{2}-}}\right)$ increase for the variation of the modified energy [5].

3. SUMMARY OF [4]

In this section we recall two results from [4] that we use in the proof of Theorem 1.1. The first one shows that if we the $L_t^4 L_x^4$ norm of a solution to (1.1) is small then we control several norms. This result will be extensively used in establishing the almost conservation law: see Proposition 4.1.

Proposition 3.1 (Modified Local Well-Posedness [4]). *Let u be a solution to (1.1). Assume that (q, r) is admissible, i.e. $(q, r) \in (2, \infty] \times [2, \infty)$ and $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. Assume also that*

$$(3.1) \quad \sup_{t \in J} E(Iu(t)) \leq 2$$

Then there exists $0 < \epsilon \ll 1$ such that if

$$(3.2) \quad \|Iu\|_{L_t^4(J)L_x^4} \leq \epsilon$$

then

$$(3.3) \quad Z(J, u) \lesssim 1$$

with

$$(3.4) \quad Z(J, u) := \sup_{(q,r) \text{ admissible}} \|\langle D \rangle Iu\|_{L_t^q(J)L_x^r}$$

The next result is a long-time estimate

Proposition 3.2 (Morawetz-type estimates [4], p9). *Let J be an interval and let (J_k) be a partition of J . Let u be the solution to (1.1) with data $u(0) = u_0 \in H^s(\mathbb{R}^2)$. Then*

$$(3.5) \quad \|Iu\|_{L_t^4(J)L_x^4}^4 \lesssim |J|^{\frac{1}{3}} \left(\sup_{t \in J} \|Iu(t)\|_{\dot{H}^1} \|Iu\|_{L^2}^3 + \|u_0\|_{L^2}^4 + \sum_k \frac{Z^6(J_k, u)}{N^{1-}} \right)$$

This inequality results from the two following estimates

$$(3.6) \quad \|Iu\|_{L_t^4(J)L_x^4}^4 \lesssim |J|^{\frac{1}{3}} \left(\sup_{t \in J} \|Iu(t)\|_{\dot{H}^1} \|Iu\|_{L^2}^3 + \|u_0\|_{L^2}^4 + \text{Error}(u, J) \right)$$

and

$$(3.7) \quad \text{Error}(u, J) \lesssim \sum_k \frac{Z^6(J_k, u)}{N^{1-}}$$

4. PROOF OF GLOBAL WELL-POSEDNESS IN $H^s(\mathbb{R}^2)$, $1 > s > \frac{1}{3}$

In this section we prove the global existence of (1.1) in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, $1 > s > \frac{1}{3}$. Our proof relies on an intermediate result that we prove in the next sections. More precisely we shall show the following

Proposition 4.1 (Almost Conservation Law in $L_t^q L_x^r$ spaces). *Let u be a solution of (1.1). Assume that (3.1) and (3.2) hold. Then*

$$(4.1) \quad \left| \sup_{t \in J} \tilde{E}(u(t)) - \tilde{E}(u(a)) \right| \lesssim \frac{1}{N^{2-}} + \frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{\theta N^{3-}}$$

For the remainder of the section we show that Proposition 4.1 implies Theorem 1.1.

Let $T > 0$, $N = N(T) \gg 1$ be a parameter to be chosen. There are three steps to prove Theorem 1.1.

(1) **Scaling.** We recall (see [5]) that there exists $C_0 := C_0(\|u_0\|_{H^s(\mathbb{R}^2)})$ such that if λ satisfies

$$(4.2) \quad \lambda = C_0 N^{\frac{1-s}{s}}$$

then

$$(4.3) \quad E(Iu^\lambda(0)) \leq \frac{1}{2}$$

with u^λ defined in (1.9).

(2) **Bootstrap.** Let F_T denote the following set

$$(4.4) \quad F_T = \left\{ T' \in [0, T] : \begin{array}{l} \sup_{t \in [0, \lambda^2 T']} E(Iu^\lambda(t)) \leq 1 \text{ and} \\ \|Iu^\lambda\|_{L_t^4(J)L_x^4}^4 \leq 2C_{mor}|J|^{\frac{1}{3}} \max(\|u_0\|_{L^2}^3, \|u_0\|_{L^2}^4, 1), J \in \mathcal{P}_{N^{3-}}([0, \lambda^2 T']) \end{array} \right\}$$

with λ defined in (4.2) and C_{mor} the constant determined by \lesssim in (3.5). We claim that F_T is the whole set $[0, T]$ for $N = N(T, \|u_0\|_{H^s(\mathbb{R}^2)}) \gg 1$ to be chosen later. Indeed

- $F_T \neq \emptyset$ since $0 \in F_T$ by (4.3)
- F_T is closed by the dominated convergence theorem.
- F_T is open. Let $\widetilde{T'} \in F_T$. Then by continuity there exists $\delta > 0$ such that for every $T' \in (\widetilde{T'} - \delta, \widetilde{T'} + \delta) \cap [0, T]$

$$(4.5) \quad \sup_{t \in [0, \lambda^2 T']} E(Iu^\lambda(t)) \leq 2$$

and

$$(4.6) \quad \|Iu^\lambda\|_{L_t^4(J)L_x^4}^4 \leq 4C_{mor}|J|^{\frac{1}{3}} \max(\|u_0\|_{L^2}^3, \|u_0\|_{L^2}^4, 1)$$

for $J \in \mathcal{P}_{N^{3-}}([0, \lambda^2 T'])$.

Let C_{fix} be the constant determined by \lesssim in (2.9).

Let $\theta = \frac{1}{N}$. Then by Proposition 2.1, (4.3) and the triangle inequality we have

$$(4.7) \quad \begin{aligned} |\tilde{E}(u^\lambda(0))| &\leq \frac{1}{2} + \frac{C_{fix}}{N^{1-}} \\ &\leq \frac{5}{8} \end{aligned}$$

Then we divide each $J = [a_j, b_j]$ into subintervals J_k , $k \in \{1, \dots, l\}$ such that $\|Iu^\lambda\|_{L_t^4(J_k)L_x^4} = \epsilon$, $k \in \{1, \dots, l-1\}$ and $\|Iu^\lambda\|_{L_t^4(J_l)L_x^4} \leq \epsilon$ with ϵ defined in Proposition 3.1. By (4.6) we have

$$(4.8) \quad l \lesssim N^{1--}$$

By Proposition 3.1, Proposition 4.1, (4.7), (4.8) and by iteration we have

$$(4.9) \quad \left| \sup_{t \in J} \tilde{E}(u^\lambda(t)) - \tilde{E}(u^\lambda(a_j)) \right| \lesssim \frac{N}{N^{2-}} \lesssim \frac{1}{N^{1-}}$$

Now we iterate again to cover $[0, \lambda^2 T']$. The number of intervals J is bounded by $\frac{\lambda^2 T}{N^{3-}}$. Therefore by this observation, (4.7) and (4.9) we have

$$(4.10) \quad \left| \sup_{t \in [0, \lambda^2 T']} \tilde{E}(u^\lambda(t)) \right| - \frac{5}{8} \lesssim \frac{\lambda^2 T}{N^{4-}}$$

By (4.10) and Proposition 2.1 we have

$$(4.11) \quad \left| \sup_{t \in [0, \lambda^2 T]} E(Iu^\lambda(t)) \right| - \frac{5}{8} \lesssim \frac{\lambda^2 T}{N^{4-}} + \frac{1}{N^{1-}}$$

Let C_{tot} be the constant determined by \lesssim in (4.11). Since $s > \frac{1}{3}$ then for every $T > 0$ we can always choose $N = N(T) \gg 1$ such that $C_{tot} \left(\frac{\lambda^2 T}{N^{4-}} + \frac{1}{N^{1-}} \right) \leq \frac{1}{8}$. Consequently $\sup_{t \in [0, \lambda^2 T']} \tilde{E}(u(t)) \lesssim 1$.

It remains to prove $\|Iu^\lambda\|_{L_t^4(J)L_x^4}^4 \leq 4C_{mor}|J|^{\frac{1}{3}} \max(\|u_0\|_{L^2}^3, \|u_0\|_{L^2}^4)$, $J \in \mathcal{P}_{N^{3-}}([0, \lambda^2 T'])$. We get from Proposition 3.2, (4.8) and the elementary inequality $\|Iu(t)\|_{L^2} \leq \|u_0\|_{L^2}$

$$(4.12) \quad \|Iu^\lambda\|_{L_t^4(J)L_x^4}^4 - C_{mor}|J|^{\frac{1}{3}} (\sqrt{2}\|u_0\|_{L^2}^3 + \|u_0\|_{L^2}^4) \leq C_{mor} \frac{|J|^{\frac{1}{3}}}{N^{+}}$$

Hence

$$(4.13) \quad \|Iu^\lambda\|_{L_t^4(J)L_x^4}^4 \leq 2C_{mor}|J|^{\frac{1}{3}} \max(\|u_0\|_{L^2}^3, \|u_0\|_{L^2}^4, 1)$$

(3) **Accounting.** Following the I - method described in [5]

$$(4.14) \quad \sup_{t \in [0, T]} E(Iu(t)) \lesssim \lambda^2 \sup_{t \in [0, \lambda^2 T]} E(Iu^\lambda(t)) \lesssim \lambda^2$$

By Proposition 1.2 we have global well-posedness of the defocusing cubic Schrödinger equation in $H^s(\mathbb{R}^2)$, $1 > s > \frac{1}{3}$. Let $T \gg 1$. Then choosing $N = N(T) \gg 1$ such that

$$(4.15) \quad \frac{0.9}{8} \leq C_{tot} \left(\frac{\lambda^2 T}{N^{4-}} + \frac{1}{N^{1-}} \right) \leq \frac{1}{8}$$

we have $N \sim T^{\frac{s}{6s-2}+}$. Plugging this value of N into (4.14) and using (1.24) we obtain (1.8).

5. PROOF OF ALMOST CONSERVATION LAW IN $L_t^q L_x^r$ SPACES

We modify an argument used in [6]. Recall that the derivative of \tilde{E} is given by the following formula

$$(5.1) \quad \partial_t \tilde{E}(u(t)) = \Lambda_4(\mu + \sigma_4 \alpha_4, u(t)) + \Lambda_6(\nu_6, u(t)).$$

Let $J = [a, b]$ be an interval included in $[0, \infty)$ and let u be such that (1.1), (3.1) and (3.2) hold. The proof of the almost conservation follows from (5.1), the quadrilinear estimate

$$(5.2) \quad |\Lambda_{4,J}(\mu + \sigma_4 \alpha_4, u)| \lesssim \frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}}$$

and the sextilinear estimate

$$(5.3) \quad |\Lambda_{6,J}(\sigma_6, u)| \lesssim \frac{1}{\theta N^{3-}}.$$

Recall that α_4 , μ and σ_4 are defined in (2.6), (2.5) and (2.8) respectively.

5.1. Proof of the quadrilinear estimate. For convenience let ν_4 denote the following multiplier

$$(5.4) \quad \nu_4 := \mu + \sigma_4 \alpha_4$$

so that by (2.5) and (2.6) we have

$$(5.5) \quad \nu_4 = \frac{i}{4} (|\xi_1|^2 m^2(\xi_1) - |\xi_2|^2 m^2(\xi_2) + |\xi_3|^2 m^2(\xi_3) - |\xi_4|^2 m^2(\xi_4)) \chi_{L^c \cap \Gamma^c}.$$

Notice that $|\Lambda_{4,J}|$ is symmetric under swapping ξ_1, ξ_2 with ξ_3, ξ_2 respectively. Therefore we may assume $|\xi_1| \geq |\xi_3|$ and $|\xi_2| \geq |\xi_4|$. Notice that if we swap ξ_1, ξ_3 with ξ_2, ξ_4 then $|\Lambda_{4,J}|$ restricted to the set $\{(\xi_1, \dots, \xi_4), |\xi_1| \geq |\xi_3|, |\xi_2| \geq |\xi_4|\}$ remains invariant. Therefore we may also assume $|\xi_3| \geq |\xi_4|$. Now we can restrict to $|\xi_1| \sim |\xi_2|$ since if not we cannot have $|\cos(\xi_{12}, \xi_{14})| \leq \theta$. Eventually it suffices to prove

$$(5.6) \quad |\Lambda_{4,J}(\nu_4 \chi_{\Sigma_4}; u)| \lesssim \frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}}$$

with

$$(5.7) \quad \Sigma_4 = \{(\xi_1, \dots, \xi_4), |\xi_1| \gtrsim N, |\xi_1| \sim |\xi_2|, |\xi_1| \geq |\xi_3| \geq |\xi_4|, |\cos(\xi_{12}, \xi_{14})| \leq \theta\}.$$

Then we need the following lemma

Lemma 5.1. *Let $A \in \Omega_k^{1 \rightarrow 4}$, $k \in \{0, \dots, 4\}$. Then*

$$(5.8) \quad |\Lambda_{4,J,A}(\nu_4 \chi_{\Sigma_4}; u)| \lesssim \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}} \right) \|\langle D \rangle I u\|_{L_t^1(d\mu^l, J) L_x^2}^k \|\langle D \rangle I(|u|^2 u)\|_{L_t^1(J) L_x^2}^{4-k}$$

Let us postpone the proof of this lemma to later and let us assume that it is true for the moment. Then we have

$$(5.9) \quad \begin{aligned} \|\langle D \rangle I(|u|^2 u)\|_{L_t^1(J) L_x^2} &\lesssim \|\langle D \rangle I u\|_{L_t^3(J) L_x^6} \left(\|P_{\ll N} u\|_{L_t^3(J) L_x^6} + \|P_{\gtrsim N} u\|_{L_t^3(J) L_x^6} \right)^2 \\ &\lesssim \|\langle D \rangle I u\|_{L_t^3(J) L_x^6} \left(\|\langle D \rangle I u\|_{L_t^3(J) L_x^6} + \frac{\|\langle D \rangle I u\|_{L_t^3(J) L_x^6}}{N} \right)^2 \\ &\lesssim Z^3(J, u) \end{aligned}$$

by the fractional Leibnitz rule and by Hölder inequality. Moreover

$$(5.10) \quad \begin{aligned} \|\langle D \rangle I u\|_{L_t^1(J, d\mu^l) L_x^2} &\lesssim \|\langle D \rangle I u\|_{L_t^\infty(J) L_x^2} \\ &\lesssim Z(J, u). \end{aligned}$$

Therefore by Proposition 3.1, Lemma 5.1, (5.9) and (5.10) we have

$$\begin{aligned}
(5.11) \quad |\Lambda_{4,J}(\nu_4 \chi_{\Sigma_4}; u)| &\leq \sum_{k=0}^4 \sum_{A \in \Omega_k^{1 \rightarrow 4}} |\Lambda_{4,J,A}(\nu_4 \chi_{\Sigma_4}; u)| \\
&\lesssim \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}} \right) \sum_{k=0}^4 \|\langle D \rangle I u\|_{L_t^1(J, d\mu^t) L_x^2}^k \|\langle D \rangle I(|u|^2 u)\|_{L_t^1(J) L_x^2}^{4-k} \\
&\lesssim \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}} \right) \sum_{k=0}^4 Z^{12-2k}(J, u) \\
&\lesssim \frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}}.
\end{aligned}$$

This proves the quadrilinear estimate (5.2).

5.1.1. *Proof of Lemma 5.1.* Given $k \in \{0, \dots, 4\}$ and $A \in \Omega_k^{1 \rightarrow 4}$ let w_j , $j \in \{1, \dots, 4\}$ denote the following functions

$$(5.12) \quad w_j(t_j) := \begin{cases} u(t_j), & j \text{ odd}, j \in A \\ |u|^2 u(t_j), & j \text{ odd}, j \notin A \\ \overline{u}(t_j), & j \text{ even}, j \in A \\ |u|^2 \overline{u}(t_j), & j \text{ even}, j \notin A \end{cases}$$

and

$$(5.13) \quad Q(t_1, \dots, t_4) := \int_{\cap_{j=1}^4 [t_j, b]} \int_{\xi_1 + \dots + \xi_4 = 0} \nu_4 \chi_{\Sigma_4} \prod_{1 \leq j \leq 4} e^{i\epsilon(j)(t-t_j)|\xi_j|^2} \widehat{w_j}(t_j, \xi_j) dt$$

with

$$(5.14) \quad \epsilon(j) = \begin{cases} 1, & j \text{ even} \\ -1, & j \text{ odd}. \end{cases}$$

We have

$$(5.15) \quad |\Lambda_{4,J,4}(\nu_4 \chi_{\Sigma_4}; u)| = \left| \int_J \int_{\xi_1 + \dots + \xi_4 = 0} \nu_4 \chi_{\Sigma_4} \left(\frac{\prod_{j \in A} \int_a^t e^{i\epsilon(j)(t-t_j)|\xi_j|^2} \widehat{w_j}(t_j, \xi_j) d\mu_l(t_j)}{\prod_{j \notin A} \int_a^t e^{i\epsilon(j)(t-t_j)|\xi_j|^2} \widehat{w_j}(t_j, \xi_j) dt_j} \right) dt \right|$$

and by Fubini

$$(5.16) \quad |\Lambda_{4,J,A}(\nu_4 \chi_{\Sigma_4}; u)| = \left| \int_{J^4} Q(t_1, \dots, t_4, w_1, \dots, w_4) \left(\prod_{j \in A} d\mu_l(t_j) \right) \left(\prod_{j \notin A} dt_j \right) \right|.$$

If we could prove

$$(5.17) \quad |Q(t_1, \dots, t_4, w_1, \dots, w_4)| \lesssim \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}} \right) \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}$$

then (5.8) would follow from (5.16) and (5.17).

We perform a Paley-Littlewood decomposition to prove (5.17). Let X denote the left-hand side of (5.17) after decomposition. By Plancherel's theorem

$$(5.18) \quad X = \left| \int_{\tau_0 + \tau' + \tau'' = 0} \widehat{\chi}_{\cap_{j=1}^4 [t_j, b]}(\tau_0) \int_{\xi' + \xi'' = 0} \nu_4 \chi_{\Sigma_4} \begin{pmatrix} \int_{\xi_1 + \xi_3 = \xi'} \chi_{|\xi_1| \sim N_1} \chi_{|\xi_3| \sim N_3} \chi_{|\cos(\xi_{12}, \xi_{14})| \leq \theta} \delta(\tau' + |\xi_1|^2 + |\xi_3|^2) \widehat{w(t_1)}(\xi_1) \widehat{w(t_3)}(\xi_3) \\ \int_{\xi_2 + \xi_4 = \xi''} \chi_{|\xi_2| \sim N_2} \chi_{|\xi_4| \sim N_4} \delta(\tau'' - |\xi_2|^2 - |\xi_4|^2) \widehat{w(t_2)}(\xi_2) \widehat{w(t_4)}(\xi_4) \end{pmatrix} \right|$$

and we want to prove

$$(5.19) \quad X \lesssim N_1^{--} N_4^+ \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}}} + \frac{1}{N^2} \right) \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}.$$

Since the L^2 - norm only depends on the magnitude of the Fourier transform we may assume that $\widehat{w(t_j)} \geq 0$, $j \in \{1, \dots, 4\}$. There are two cases:

- **Case 1:** $N_3 \gg N_4$. Recall (see [6]) that

$$(5.20) \quad |\cos(\xi_1, \xi_3)| \lesssim \theta_0 + \frac{N_4}{N_3}$$

and

$$(5.21) \quad \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \leq m^2(N_1) N_1 N_3 \theta + m^2(N_3) N_3^2$$

There are two subcases

- **Case 1.a:** $\theta \gtrsim \frac{N_4}{N_3}$
We have

$$(5.22) \quad \left| \widehat{\chi}_{\cap_{j=1}^4 [t_j, b]}(\tau_0) \right| \lesssim \tau_0$$

We introduce the logarithmic weight

$$(5.23) \quad q(\tau) := 1 + \log^2(\tau).$$

Notice that $q(\tau' + \tau'') \lesssim q(\tau') + q(\tau'')$. Let

$$(5.24) \quad \widetilde{B_{\epsilon, 1, 3}}(\tau', \xi') := \int_{\xi_1 + \xi_3 = \xi'} \chi_{|\xi_1| \sim N_1} \chi_{|\xi_3| \sim N_3} \chi_{[-\epsilon, \epsilon]}(\tau' - |\xi_1|^2 - |\xi_3|^2) \chi_{|\cos(\xi_1, \xi_3)| \lesssim \theta_0} \widehat{w(t_1)}(\xi_1) \widehat{w(t_3)}(\xi_3)$$

and

$$(5.25) \quad \widetilde{B_{\epsilon, 2, 4}}(\tau'', \xi'') := \int_{\xi_2 + \xi_4 = \xi''} \chi_{|\xi_2| \sim N_2} \chi_{|\xi_4| \sim N_4} \chi_{[-\epsilon, \epsilon]}(\tau'' - |\xi_2|^2 - |\xi_4|^2) \widehat{w(t_2)}(\xi_2) \widehat{w(t_4)}(\xi_4).$$

Then by Hausdorff-Young, (2.13), (2.14) and (5.22)

$$\begin{aligned}
(5.26) \quad X &\lesssim q^2(N_1) \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \lim_{\epsilon \rightarrow 0} \frac{1}{(2\epsilon)^2} \int_{\mathbb{R}} \frac{1}{\langle \tau_0 \rangle q(\tau_0)} \left(\widetilde{B_{\epsilon,1,3}} * \widetilde{B_{\epsilon,2,4}}(\tau_0, 0) \right) d\tau_0 \\
&\lesssim q^2(N_1) \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \overline{\lim} \frac{1}{(2\epsilon)^2} \|\widetilde{B_{\epsilon,1,3}} * \widetilde{B_{\epsilon,2,4}}\|_{L_{\tau_0}^\infty L_{\xi_0}^\infty} \\
&\lesssim q^2(N_1) \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \overline{\lim} \frac{1}{(2\epsilon)^2} \|B_{\epsilon,1,3} B_{\epsilon,2,4}\|_{L_t^1 L_x^1} \\
&\lesssim q^2(N_1) \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \overline{\lim} \frac{1}{(2\epsilon)^2} \|B_{\epsilon,1,3}\|_{L_t^2 L_x^2} \|B_{\epsilon,2,4}\|_{L_t^2 L_x^2} \\
&\lesssim q^2(N_1) \frac{m^2(N_1) N_1 N_3 \theta + m^2(N_3) N_3^2}{\langle N_1 \rangle m(N_1) \dots \langle N_4 \rangle m(N_4)} \theta^{\frac{1}{2}} \left(\frac{N_4}{N_2} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2} \\
&\lesssim N_1^{-} N_4^{+} \frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}
\end{aligned}$$

– **Case 1.b:** $\theta \ll \frac{N_4}{N_3}$

In this case $|\cos(\xi_1, \xi_3)| \lesssim \frac{N_4}{N_3}$ and

$$\begin{aligned}
(5.27) \quad X &\lesssim q^2(N_1) \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \overline{\lim} \frac{1}{(2\epsilon)^2} \|B_{\epsilon,1,3} B_{\epsilon,2,4}\|_{L_t^1 L_x^1} \\
&\lesssim q^2(N_1) \frac{m^2(N_1) N_1 N_3 \theta + m^2(N_3) N_3^2}{\langle N_1 \rangle m(N_1) \dots \langle N_4 \rangle m(N_4)} \left(\frac{N_4}{N_3} \right)^{\frac{1}{2}} \left(\frac{N_4}{N_2} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2} \\
&\lesssim N_1^{-} N_4^{+} \left(\frac{\theta^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{2-}} \right) \prod_{j=1}^6 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}
\end{aligned}$$

with

$$(5.28) \quad \widetilde{B_{\epsilon,1,3}}(\tau', \xi') := q(\tau') \int_{\xi_1 + \xi_3 = \xi'} \chi_{|\xi_1| \sim N_1} \chi_{|\xi_3| \sim N_3} \chi_{[-\epsilon, \epsilon]}(\tau' - |\xi_1|^2 - |\xi_3|^2) \chi_{|\cos(\xi_1, \xi_3)| \lesssim \frac{N_4}{N_3}} \widehat{w(t_1)}(\xi_1) \widehat{w(t_3)}(\xi_3)$$

and $\widetilde{B_{\epsilon,2,4}}$ defined in (5.25).

• **Case 2:** $N_3 \sim N_4$. Recall (see [6]) that

$$(5.29) \quad \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \lesssim m^2(N_1) N_1 N_3.$$

We have

$$\begin{aligned}
(5.30) \quad X &\lesssim q^2(N_1) \|\nu_4 \chi_{\Sigma_4}\|_{L^\infty} \overline{\lim} \frac{1}{(2\epsilon)^2} \|B_{\epsilon,1,3} B_{\epsilon,2,4}\|_{L_t^1 L_x^1} \\
&\lesssim q^2(N_1) \frac{m^2(N_1) N_1 N_3}{\langle N_1 \rangle m(N_1) \dots \langle N_4 \rangle m(N_4)} \left(\frac{N_3}{N_1} \right)^{\frac{1}{2}} \left(\frac{N_4}{N_2} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2} \\
&\lesssim N_1^{-} N_4^{+} \frac{1}{N^{2-}} \prod_{j=1}^4 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}
\end{aligned}$$

with

$$\begin{aligned}
(5.31) \quad \widetilde{B_{\epsilon,1,3}}(\tau', \xi') &:= q(\tau') \int_{\xi_1 + \xi_3 = \xi'} \chi_{|\xi_1| \sim N_1} \chi_{|\xi_3| \sim N_3} \chi_{[-\epsilon, \epsilon]}(\tau' + |\xi_1|^2 + |\xi_3|^2) \widehat{w_1(t_1)}(\xi_1) \widehat{w_3(t_3)}(\xi_3) \\
&\text{and } \widetilde{B_{\epsilon,2,4}} \text{ defined in (5.25).}
\end{aligned}$$

5.2. Proof of the sextilinear estimate. Notice that $\nu_6 = 0$ if $\max(|\xi_1|, \dots, |\xi_6|) \ll N$. Let $|\xi_{1*}| \geq \dots \geq |\xi_{6*}|$ be the six amplitudes in order. The convolution constraint $\xi_1 + \dots + \xi_6 = 0$ imposes $|\xi_{1*}| \sim |\xi_{2*}|$. It suffices to prove

$$(5.32) \quad |\Lambda_{6,J}(\nu_6 \chi_{\Sigma_6}, u)| \lesssim \frac{1}{\theta N^{3-}}$$

with

$$(5.33) \quad \Sigma_6 = \{(\xi_1, \dots, \xi_6), |\xi_{1*}| \gtrsim N, |\xi_{1*}| \sim |\xi_{2*}|\}.$$

We will prove the following lemma

Lemma 5.2. *Let $A \in \Omega_k^{1 \rightarrow 6}$, $k \in \{0, \dots, 6\}$. Then*

$$(5.34) \quad |\Lambda_{6,J,A}(\nu_6 \chi_{\Sigma_6}; u)| \lesssim \frac{1}{\theta N^{3-}} \|\langle D \rangle I u\|_{L_t^1(d\mu^t, J) L_x^2}^k \|\langle D \rangle I(|u|^2 u)\|_{L_t^1(J) L_x^2}^{6-k}$$

Assuming that it is true then by (5.9), (5.10) and Proposition 3.1 we have

$$(5.35) \quad \begin{aligned} |\Lambda_{6,J}(\nu_6 \chi_{\Sigma_6}; u)| &\leq \sum_{k=0}^6 \sum_{A \in \Omega_k^{1 \rightarrow 6}} |\Lambda_{6,J,A}(\nu_6 \chi_{\Sigma_6}; u)| \\ &\lesssim \frac{1}{\theta N^{3-}} \sum_{k=0}^6 \|\langle D \rangle I u\|_{L_t^1(J, d\mu^t) L_x^2}^k \|\langle D \rangle I(|u|^2 u)\|_{L_t^1(J) L_x^2}^{6-k} \\ &\lesssim \frac{1}{\theta N^{3-}} \sum_{k=0}^6 Z^{18-2k}(J, u) \\ &\lesssim \frac{1}{N^{3-}} \end{aligned}$$

which proves the sextilinear estimates.

5.2.1. *Proof of Lemma 5.2.* Given $k \in \{0, \dots, 6\}$ and $A \in \Omega_k^{1 \rightarrow 6}$ let w_j , $j \in \{1, \dots, 6\}$ denote the following functions

$$(5.36) \quad w_j(t_j) := \begin{cases} u(t_j), & j \text{ odd}, j \in A \\ |u|^2 u(t_j), & j \text{ even}, j \notin A \\ \overline{u}(t_j), & j \text{ even}, j \in A \\ |u|^2 \overline{u}(t_j), & j \text{ even}, j \notin A \end{cases}$$

and

$$(5.37) \quad Q(t_1, \dots, t_6; w_1, \dots, w_6) := \int_{\cap_{j=1}^6 [t_j, b]} \int_{\xi_1 + \dots + \xi_6 = 0} \nu_6 \chi_{\Sigma_6} \prod_{1 \leq j \leq 6} e^{i\epsilon(j)(t-t_j)|\xi_j|^2} \widehat{w_j}(t_j, \xi_j) dt$$

with $\epsilon(j)$ defined in (5.14). We have

$$(5.38) \quad |\Lambda_{6,J,A}(\nu_6; u)| = \left| \int_J \int_{\xi_1 + \dots + \xi_6 = 0} \nu_6 \left(\frac{\left(\prod_{j \in A} \int_a^t e^{i\epsilon(j)(t-t_j)|\xi_j|^2} \widehat{w_j}(t_j, \xi_j) d\mu_l(t_j) \right)}{\left(\prod_{j \notin A} \int_a^t e^{i\epsilon(j)(t-t_j)|\xi_j|^2} \widehat{w_j}(t_j, \xi_j) dt_j \right)} \right) dt \right|$$

and by Fubini

$$(5.39) \quad |\Lambda_{6,J,A}(\nu_6 \chi_{\Sigma_6}; u)| = \left| \int_{J^6} Q(t_1, \dots, t_6, w_1, \dots, w_6) \left(\prod_{j \in A} d\mu_l(t_j) \right) \left(\prod_{j \notin A} dt_j \right) \right|.$$

If we could prove

$$(5.40) \quad |Q(t_1, \dots, t_6, w_1, \dots, w_6)| \lesssim \frac{1}{\theta N^{3-}} \prod_{j=1}^6 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}$$

then (5.34) would follow from (5.39) and (5.40). It remains to show (5.40). By decomposition we may assume $\widehat{w_j}(t_j) \geq 0$.

We perform a Paley-Littlewood decomposition to prove (5.40). Let X be the left-hand side of (5.40). By Plancherel we have

$$(5.41) \quad X = \left| \int_{\tau=0} \widehat{\chi_{\cap_{j=1}^6 [t_j, b]}}(\tau_0) \int_{\xi=0} \nu_6 \left[\begin{aligned} &\left(\int_{\xi_1^* + \xi_3^* = \xi'} \left[\begin{aligned} &\chi_{|\xi_1^*| \sim N_1^*} \chi_{|\xi_3^*| \sim N_3^*} \\ &\delta(\tau' \pm |\xi_1^*|^2 \pm |\xi_3^*|^2) \widehat{w_1}(t_1^*, \xi_1^*) \widehat{w_3}(t_3^*, \xi_3^*) \end{aligned} \right] \right) \\ &\left(\int_{\xi_2^* + \xi_4^* = \xi''} \left[\begin{aligned} &\chi_{|\xi_2^*| \sim N_2^*} \chi_{|\xi_4^*| \sim N_4^*} \\ &\delta(\tau'' \pm |\xi_2^*|^2 \pm |\xi_4^*|^2) \widehat{w_2}(t_2^*, \xi_2^*) \widehat{w_4}(t_4^*, \xi_4^*) \end{aligned} \right] \right) \\ &\chi_{|\xi_5^*| \sim N_5^*} \delta(\tau_5^* \pm |\xi_5^*|^2) \widehat{w_5}(t_5^*, \xi_5^*) \\ &\chi_{|\xi_6^*| \sim N_6^*} \delta(\tau_6^* \pm |\xi_6^*|^2) \widehat{w_6}(t_6^*, \xi_6^*) \end{aligned} \right] \right|$$

where $N_{1^*} \geq \dots \geq N_{6^*}$ are the dyadic numbers in order, $1^*, \dots, 6^*$ are the corresponding subscripts, $*_{\tau} := \tau_0 + \tau' + \tau'' + \tau_{5^*} + \tau_{6^*}$, $*_{\xi} := \xi' + \xi'' + \xi_{5^*} + \xi_{6^*}$, $\pm|\xi_j|^2$ denotes $+|\xi_j|^2$ if j is odd and $-|\xi_j|^2$ if j is even. We would like to prove

$$(5.42) \quad X \lesssim \frac{N_{1^*}^{--} N_{6^*}^{++}}{\theta N^{3--}} \prod_{j=1}^6 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}$$

Again we can assume the $\widehat{w_j}(t_j) \geq 0$. Notice that

$$(5.43) \quad \widehat{\chi_{\cap_{j=1}^6 [t_j, b]}}(\tau_0) \lesssim \langle \tau_0 \rangle^{-1}.$$

Recall also (see [6]) that $|\nu_4(\xi_1, \dots, \xi_4)| \lesssim \frac{\min(m(N_1), \dots, m(N_4))^2}{\theta}$. Therefore

$$(5.44) \quad |\nu_6| \lesssim \sum_{k=1}^4 \frac{|\min(m^2(\xi_1), \dots, m^2(\xi_k + \dots + \xi_{k+2}), \dots, m^2(\xi_6))|}{\theta} \lesssim \frac{m^2(N_{4^*})}{\theta}.$$

Before continuing we define $M_{\epsilon, j}$ and P_{N_j} such that

$$(5.45) \quad \widetilde{M_{\epsilon, j}}(\tau_j, \xi_j) := \chi_{[-\epsilon, \epsilon]}(\tau_j \pm |\xi_j|^2) \chi_{|\xi_j| \sim N_j} \widehat{w_j}(t_j, \xi_j)$$

and

$$(5.46) \quad \widehat{P_{N_j}(f)}(\xi_j) := \chi_{|\xi_j| \sim N_j} \widehat{f}(\xi_j)$$

for $j \in [1, \dots, 6]$. Also let B_{ϵ, k^*, l^*} be such that

$$(5.47) \quad \widetilde{B_{\epsilon, k^*, l^*}}(\tau', \epsilon') := \int_{\xi_{k^*} + \xi_{l^*} = \xi'} \frac{\chi_{|\xi_{k^*}| \sim N_{k^*}} \chi_{|\xi_{l^*}| \sim N_{l^*}} \chi_{[-\epsilon, \epsilon]}(\tau' \pm |\xi_{k^*}|^2 \pm |\xi_{l^*}|^2)}{w_{k^*}(t_{k^*})(\xi_{k^*}) \overline{w_{l^*}}(t_{l^*})(\xi_{l^*})}.$$

Then we prove the following claim.

Claim: If $N_{k^*} \leq N_{l^*}$ then

$$(5.48) \quad \overline{\lim} \frac{1}{2\epsilon} \|B_{\epsilon, k^*, l^*}\|_{L_t^2 L_x^2} \lesssim \left(\frac{N_{k^*}}{N_{l^*}} \right)^{\frac{1}{2}} \|w_{k^*}(t_{k^*})\|_{L^2} \|w_{l^*}(t_{l^*})\|_{L^2}.$$

Proof. If k^* and l^* are of the same parity then the claim follows from Proposition 2.14. It remains to study the case where k^* and l^* are of different parity. Let $B_{+, k, l}$, $B_{-, k, l}$ be such that

$$(5.49) \quad \widetilde{B_{+, \epsilon, k^*, l^*}}(\tau', \epsilon') := \int_{\xi_{k^*} + \xi_{l^*} = \xi'} \frac{\chi_{|\xi_{k^*}| \sim N_{k^*}} \chi_{|\xi_{l^*}| \sim N_{l^*}} \chi_{[-\epsilon, \epsilon]}(\tau' + |\xi_{k^*}|^2 + |\xi_{l^*}|^2)}{w_{k^*}(t_{k^*})(\xi_{k^*}) \overline{w_{l^*}}(t_{l^*})(\xi_{l^*})}$$

and

$$(5.50) \quad \widetilde{B_{-, \epsilon, k^*, l^*}}(\tau', \epsilon') := \int_{\xi_{k^*} + \xi_{l^*} = \xi'} \frac{\chi_{|\xi_{k^*}| \sim N_{k^*}} \chi_{|\xi_{l^*}| \sim N_{l^*}} \chi_{[-\epsilon, \epsilon]}(\tau' - |\xi_{k^*}|^2 - |\xi_{l^*}|^2)}{w_{k^*}(t_{k^*})(\xi_{k^*}) \overline{w_{l^*}}(t_{l^*})(\xi_{l^*})}.$$

Observe that

$$\begin{aligned}
(5.51) \quad \overline{\lim} \frac{1}{2\epsilon} \|B_{\epsilon, k^*, l^*}\|_{L_t^2 L_x^2} &= \|P_{N_{k^*}}(e^{it\epsilon(k^*)\Delta} w_{k^*}(t_{k^*})) P_{N_{l^*}}(e^{it\epsilon(l^*)\Delta} w_{l^*}(t_{l^*}))\|_{L_t^2 L_x^2} \\
&= \|P_{N_{k^*}}(e^{it\epsilon(k^*)\Delta} w_{k^*}(t_{k^*})) \overline{P_{N_{l^*}}(e^{it\epsilon(l^*)\Delta} w_{l^*}(t_{l^*}))}\|_{L_t^2 L_x^2} \\
&\lesssim \overline{\lim} \frac{1}{2\epsilon} \left(\|B_{+, \epsilon, k^*, l^*}\|_{L_\tau^2 L_\epsilon^2} + \|B_{-, \epsilon, k^*, l^*}\|_{L_\tau^2 L_\epsilon^2} \right) \\
&\lesssim \left(\frac{N_{k^*}}{N_{l^*}} \right)^{\frac{1}{2}} \|w_{k^*}(t_{k^*})\|_{L^2} \|\overline{w_{l^*}(t_{l^*})}\|_{L^2} \\
&\lesssim \left(\frac{N_{k^*}}{N_{l^*}} \right)^{\frac{1}{2}} \|w_{k^*}(t_{k^*})\|_{L^2} \|w_{l^*}(t_{l^*})\|_{L^2}
\end{aligned}$$

This ends the proof of the claim. \square

Observe also that

$$\begin{aligned}
(5.52) \quad \overline{\lim} \frac{1}{2\epsilon} \|M_{\epsilon, j}\|_{L_t^\infty L_x^\infty} &\lesssim \|e^{it\pm\Delta} (P_{N_j} w_j(t_j))\|_{L_t^\infty L_x^\infty} \\
&\lesssim \frac{N_j^{\frac{3}{2}}}{m(N_j) < N_j} \|\langle D \rangle I w_j(t_j)\|_{L_x^2}
\end{aligned}$$

by Plancherel and Bernstein inequalities.

By (5.43), (5.44), (5.52), the claim and Hausdorff-Young we have

$$\begin{aligned}
(5.53) \quad X &\lesssim \frac{m^2(N_{4^*})}{\theta} q^4(N_{1^*}) \overline{\lim} \frac{1}{(2\epsilon)^4} \|B_{\epsilon, 1^*, 3^*} * B_{\epsilon, 2^*, 4^*} * M_{\epsilon, 5^*} * M_{\epsilon, 6^*}\|_{L_\tau^\infty L_\xi^\infty} \\
&\lesssim \frac{m^2(N_{4^*})}{\theta} q^4(N_{1^*}) \overline{\lim} \frac{1}{(2\epsilon)^4} \|B_{\epsilon, 1^*, 3^*} B_{\epsilon, 2^*, 4^*} M_{\epsilon, 5^*} M_{\epsilon, 6^*}\|_{L_t^1 L_x^1} \\
&\lesssim q^4(N_{1^*}) \frac{m^2(N_{4^*})}{\theta} \overline{\lim} \frac{1}{(2\epsilon)^4} \|B_{\epsilon, 1^*, 3^*}\|_{L_t^2 L_x^2} \|B_{\epsilon, 2^*, 4^*}\|_{L_t^2 L_x^2} \|M_{\epsilon, 5^*}\|_{L_t^\infty L_x^\infty} \|M_{\epsilon, 6^*}\|_{L_t^\infty L_x^\infty} \\
&\lesssim q^4(N_{1^*}) \frac{m^2(N_{4^*})}{\theta < N_{1^*} > m(N_{1^*}) \dots < N_{4^*} > m(N_{4^*})} \left(\frac{N_{3^*}}{N_{1^*}} \right)^{\frac{1}{2}} \left(\frac{N_{4^*}}{N_{2^*}} \right)^{\frac{1}{2}} N_{5^*}^{\frac{3}{2}} N_{6^*}^{\frac{3}{2}} \prod_{j=1}^6 \|\langle D \rangle I w_j(t_j)\|_{L_x^2} \\
&\lesssim \frac{N_{1^*}^- - N_{6^*}^+}{\theta N^{3-}} \prod_{j=1}^6 \|\langle D \rangle I w_j(t_j)\|_{L_x^2}
\end{aligned}$$

with q being the logarithmic weight introduced in (5.23).

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